Blacktown Boys' High School

2024

HSC Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided for this paper
- All diagrams are not drawn to scale
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7 – 12)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Assessor: X. Chirgwin

Student Name: ______________________________

Teacher Name: ______________________________

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 2024 Higher School Certificate Examination.

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Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1–10.

Q1 Consider the following statement.

‘If Steve is stressed, then he does not sleep well.”

Which of the following is the converse of this statement?

A. If Steve is not stressed, then he does not sleep well.

B. If Steve is stressed, then he does sleep well.

C. If Steve does not sleep well, then he is stressed.

D. If Steve does sleep well, then he is not stressed.

Q2 The line $L = \begin{pmatrix} 9 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ forms an angle with the positive x-axis.

What is the size of this angle?

A. $12.5°$

B. $57.7°$

C. $74.5°$

D. $143.3°$

Q3 Which expression is equal to $\int 2\tan^3 x \sec^2 x \, dx$?

A. $2\tan^4 x + C$

B. $\dfrac{\tan^4 x}{2} + C$

C. $2\tan^5 x + 2\tan^3 x + C$

D. $\dfrac{2\tan^5 x}{5} + \dfrac{2\tan^3 x}{3} + C$

Q4 The polynomial $P(z) = (z-a)(z-b)(z-c)$ has complex roots a, b and c, where $Re(a) \neq 0, Re(b) \neq 0, Re(c) \neq 0$ and $Im(b) = 0$. When expanded, the polynomial is a cubic with real coefficients. Which one of the following statements is definitely true?

A. $|a| = |c|$

B. $a + c = 0$

C. $|a| = |b|$

D. $a - c = 0$

Q5 A particle undergoes simple harmonic motion about the origin. Its displacement in centimetre is given by

$$x = 5\cos\left(\frac{t}{2} + \frac{\pi}{3}\right)$$

The particle is at rest when

A. $t = 0$

B. $t = \dfrac{\pi}{3}$

C. $t = \dfrac{2\pi}{3}$

D. $t = \dfrac{4\pi}{3}$

Q6 A 14 kilogram mass moves in a straight line under the action of a variable force F, so that its velocity $v\ ms^{-1}$ when it is x metres from the origin is given by $v = \sqrt{5x - x^3 + 10}$. The force F acting on the mass is given by

A. $14\sqrt{5x - x^3 + 10}$

B. $14(5x^2 - x^4 + 10x)$

C. $7(5 - 3x^2)$

D. $7(5x - x^3 + 10)$

Q7 Which expression is equal to $\displaystyle\int_{-3}^{a} \frac{dx}{\sqrt{7 - 6x - x^2}}$?

A. $\sin^{-1}\left(\dfrac{a-3}{2}\right)$

B. $\sin^{-1}\left(\dfrac{a-3}{4}\right)$

C. $\sin^{-1}\left(\dfrac{a+3}{2}\right)$

D. $\sin^{-1}\left(\dfrac{a+3}{4}\right)$

Q8 What is the simplest expression of

$$\text{Arg}\left(\frac{1}{1+i}\right) + \text{Arg}\left(\frac{1}{(1+i)^2}\right) + \text{Arg}\left(\frac{1}{(1+i)^3}\right) + \cdots + \text{Arg}\left(\frac{1}{(1+i)^{20}}\right)?$$

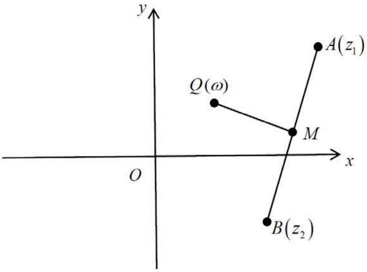
A. $-\dfrac{\pi}{4}$

B. $-\dfrac{\pi}{2}$

C. $\dfrac{\pi}{4}$

D. $\dfrac{\pi}{2}$

Q9 On the Argand diagram below, points A and B correspond to the complex numbers z_1 and z_2 respectively. Given that M is the midpoint of the interval AB and QM is drawn perpendicular to AB, and $QM = AM = BM$. If Q corresponds to the complex number ω, what is the correct expression for ω?



A. $\dfrac{z_1 + z_2}{2} + i\left(\dfrac{z_1 - z_2}{2}\right)$

B. $\dfrac{z_1 + z_2}{2} + i\left(\dfrac{z_1 + z_2}{2}\right)$

C. $i\left(\dfrac{z_1 - z_2}{2}\right)$

D. $i\left(\dfrac{z_1 + z_2}{2}\right)$

Q10 Without evaluating the integrals, which of the following equals to zero?

A. $\displaystyle\int_{-2}^{2} |x^2 - 10| dx$

B. $\displaystyle\int_{-1}^{1} (x^2 - 1)(1 - x^2)^5 dx$

C. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos^3 x + 1}{\cos^2 x} dx$

D. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^3 x \cos x}{2} dx$

End of Section I

Section II

90 Marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question on a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) The complex numbers $z = 2e^{\frac{i\pi}{4}}$ and $w = 3e^{\frac{5i\pi}{6}}$ are given. **2**

Find the value of zw^3, give the answer in the form $re^{i\theta}$.

(b) Find the quadratic equation with the roots $\sqrt{5}\,\text{cis}\left(\frac{\pi}{6}\right), \sqrt{5}\,\text{cis}\left(-\frac{\pi}{6}\right)$. **2**

(c) Find a vector equation of the line through the point $A(-4, 3, -1)$ and $B(-9, 7, 5)$. **2**

(d) Evaluate

(i) $\displaystyle\int_0^1 \frac{e^x}{e^{2x}+9}\,dx$ **2**

(ii) $\displaystyle\int_0^2 \frac{6x-4}{(x+1)(3x^2+2)}\,dx$ **4**

(iii) $\displaystyle\int_0^5 \sqrt{\frac{10-x}{10+x}}\,dx$ **3**

End of Questions 11

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) A particle moves in simple harmonic motion described by the equation $\ddot{x} = -36(x + 8)$. Find the period and the central point of motion. **2**

(b) Given that vectors $\underset{\sim}{a} = 2\underset{\sim}{i} + m\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = m^2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ are perpendicular to each other. Find the values of m. **2**

(c) Consider the statement:

$$\forall x \in \mathbb{R},\ (x > 1) \Rightarrow (x < x^2)$$

(i) Write the contrapositive. **1**

(ii) Write the negation. **1**

(d) The acceleration, a ms^{-2}, of a particle moving in a straight line given by $a = \dfrac{v}{\log_e v}$, where v is the velocity of the particle in m/s at time t seconds. The initial velocity of the particle was 5 ms^{-1}. Find the velocity of the particle in terms of t. **3**

(e) Using the substitution $t = \tan\dfrac{\theta}{2}$ to find $\displaystyle\int \frac{d\theta}{2 + 2\sin\theta + 2\cos\theta}$ **3**

(f) Use the method of proof by contradiction to prove that the cube root of any prime numbers p, where $p \geq 2$, is irrational. **3**

End of Questions 12

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) Solve the equation $x^4 - 2x^3 - 14x^2 + 78x + 225 = 0$ over the complex field given that it has a rational zero of multiplicity 2. **4**

(b) The speed v cm/sec of a particle moving with simple harmonic motion in a straight line is given by $v^2 = 176x - 936 - 8x^2$, where x cm is the magnitude of the displacement from a fixed point O.

(i) Show that $\dfrac{d^2x}{dt^2} = -8(x - 11)$. **1**

(ii) Find the amplitude of the motion. **2**

(c) (i) Prove $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **1**

(ii) Hence, show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x}dx = \frac{\pi}{4}$ **3**

(d) The straight lines l_1 and l_2 have the following vector equations

$$\underset{\sim}{r_1} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k} + \lambda\left(\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}\right)$$
$$\underset{\sim}{r_2} = 8\underset{\sim}{i} + 8\underset{\sim}{j} + 13\underset{\sim}{k} + \mu\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}\right)$$

where λ and μ are scalar parameters.

(i) Show that l_1 and l_2 intersect at some point P and find its coordinates. **2**

(ii) Calculate the acute angle between l_1 and l_2. **2**

End of Questions 13

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) Find the primitive function of $f(x) = \dfrac{x+8}{\sqrt{24-10x-x^2}}$. **3**

(b) Prove that the sum of the squares of 3 consecutive positive integers is always 1 less than a multiple of 3. **2**

(c) Evaluate $\displaystyle\int_1^2 \frac{\log_e x}{x^3}\,dx$. **4**

(d) (i) Let x and y be real numbers such that $x \geq 0$ and $y \geq 0$. **1**

Prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$.

(ii) Suppose that a, b, c are real numbers. **2**

Prove that $a^8 + b^8 + c^8 \geq a^4b^4 + a^4c^4 + b^4c^4$.

(iii) Show that $a^4b^4 + a^4c^4 + b^4c^4 \geq a^4b^2c^2 + b^4a^2c^2 + c^4a^2b^2$. **2**

(iv) Deduce that if $a^2 + b^2 + c^2 = d^2$, then $a^8 + b^8 + c^8 \geq a^2b^2c^2d^2$. **1**

End of Questions 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) A particle moves in a straight line. The displacement function x metres in terms of velocity v m/s is given by $x = 8v - 3\ln(v+2)$. Find the time t seconds of the particle as a function of its velocity v, if the particle is initially moving at 1 m/s. **4**

(b) (i) Using integration by parts to show that: **2**

$$\int_0^1 x^{1000}(1-x)^n dx = \frac{n}{1001}\int_0^1 x^{1001}(1-x)^{n-1} dx$$

(ii) If $I_n = \int_0^1 x^{1000}(1-x)^n dx,$ use part (i) to show that **2**

$$I_n = \frac{n}{1001+n} I_{n-1}$$

(iii) Hence show that $I_n = \dfrac{n!\,1000!}{(1001+n)!}$ **2**

(c) Consider the function $f(x) = \dfrac{(n+1+x)^{n+1}}{(n+x)^n}, x \geq 0$ where $n \geq 1$ is a fixed positive integer.

(i) Show that for $x > 0$, $f(x)$ is an increasing function. **2**

(ii) Hence show that $\left(1 + \dfrac{x}{n+1}\right)^{n+1} > \left(1 + \dfrac{x}{n}\right)^n$. **2**

(iii) Deduce that $(n+2)^{n+1} n^n > (n+1)^{2n+1}$. **1**

End of Questions 15

Question 16 (15 marks) Use the Question 16 Writing Booklet.

(a) Given that $P(x) = x^6 + x^3 + 1$.

(i) Show that the roots of $P(x) = 0$ are amongst the roots of $x^9 - 1 = 0$. **1**

(ii) Show that **4**

$$P(x) = \left(x^2 - 2x\cos\frac{2\pi}{9} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{9} + 1\right)\left(x^2 - 2x\cos\frac{8\pi}{9} + 1\right)$$

(iii) Evaluate $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} + \cos\frac{4\pi}{9}\cos\frac{8\pi}{9} + \cos\frac{2\pi}{9}\cos\frac{8\pi}{9}$ **2**

(b) If a function $f(x)$ is continuous for $a \le x \le b$ and

(i) Show that $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$ **1**

(ii) Hence prove that $\left|\int_0^\pi 27^x \cos x\, dx\right| \le \frac{3^{3\pi} - 1}{3\ln 3}$ **3**

(c) Given that $x_1 + x_2 > \sqrt{x_1 x_2}$, use the method of mathematical induction to show that for all positive integers $n \ge 2$, if $x_j > 1$, $j = 1, 2, 3, \dots, n$, then **4**

$$\ln(x_1 + x_2 + \cdots + x_n) > \frac{1}{2^{n-1}}(\ln x_1 + \ln x_2 + \cdots + \ln x_n)$$

End of Paper

Student Name:____

Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ~~●~~ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word 'correct' and drawing an arrow as follows.

correct

A ~~●~~ B ~~●~~ C ○ D ○

Start Here →

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

2024 Mathematics Extension 2 AT4 Trial Solutions

Section 1		
Q1	**C** $X \Longrightarrow Y$, the converse is $Y \Longrightarrow X$	1 Mark
Q2	**B** $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 2$ $\cos\theta = \dfrac{2}{\sqrt{2^2 + (-3)^2 + 1^2} \times \sqrt{1^2}}$ $\theta = 57.7°$	1 Mark
Q3	**B** $I = \int 2\tan^3 x \sec^2 x\, dx$ Let $u = \tan x$ $du = \sec^2 x\, dx$ $I = \int 2u^3 du$ $I = \dfrac{2u^4}{4} + C$ $I = \dfrac{\tan^4 x}{2} + C$	1 Mark
Q4	**A** Since $Im(b) = 0$, then b is a real root. Roots appears in conjugate pair for real polynomials. If a is a root, then c must be its conjugate pair. $a = x + iy$, then $c = x - iy$ $\|a\| = \|c\|$	1 Mark
Q5	**D** $x = 5\cos\left(\dfrac{t}{2} + \dfrac{\pi}{3}\right)$ $v = -\dfrac{5}{2}\sin\left(\dfrac{t}{2} + \dfrac{\pi}{3}\right)$ Particle at rest when $v = 0$ $\sin\left(\dfrac{t}{2} + \dfrac{\pi}{3}\right) = 0$ $\dfrac{t}{2} + \dfrac{\pi}{3} = 0, \pi, ...$ $\dfrac{t}{2} = -\dfrac{\pi}{3}, \dfrac{2\pi}{3}, ...$ $t = -\dfrac{2\pi}{3}, \dfrac{4\pi}{3}, ...$	1 Mark
Q6	**C** $v = \sqrt{5x - x^3 + 10}$ $\dfrac{1}{2}v^2 = \dfrac{1}{2}(5x - x^3)$ $a = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \dfrac{1}{2}(5 - 3x^2)$ $F = ma$ $F = 14 \times \dfrac{1}{2}(5 - 3x^2)$ $F = 7(5 - 3x^2)$	1 Mark

Q7	**D** $\int_{-3}^{a} \frac{dx}{\sqrt{7-6x-x^2}}$ $= \int_{-3}^{a} \frac{dx}{\sqrt{16-(x^2+6x+9)}}$ $= \int_{-3}^{a} \frac{dx}{\sqrt{16-(x+3)^2}}$ $= \left[\sin^{-1}\left(\frac{x+3}{4}\right)\right]_{-3}^{a}$ $= \sin^{-1}\left(\frac{a+3}{4}\right) - \sin^{-1}\left(\frac{-3+3}{4}\right)$ $= \sin^{-1}\left(\frac{a+3}{4}\right)$	1 Mark
Q8	**B** $\text{Arg}\left(\frac{1}{1+i}\right) = \text{Arg}(1) - \text{Arg}(1+i) = 0 - \frac{\pi}{4} = -\frac{\pi}{4}$ $\text{Arg}\left(\frac{1}{(1+i)^2}\right) = \text{Arg}(1) - \text{Arg}(1+i)^2 = 0 - 2\times\frac{\pi}{4} = -\frac{2\pi}{4}$ … $\text{Arg}\left(\frac{1}{(1+i)^{20}}\right) = \text{Arg}(1) - \text{Arg}(1+i)^{20} = 0 - 20\times\frac{\pi}{4} = -\frac{20\pi}{4}$ $\text{Arg}\left(\frac{1}{1+i}\right) + \text{Arg}\left(\frac{1}{(1+i)^2}\right) + \text{Arg}\left(\frac{1}{(1+i)^3}\right) + \cdots + \text{Arg}\left(\frac{1}{(1+i)^{20}}\right)$ $= -\frac{\pi}{4} - \frac{2\pi}{4} - \frac{3\pi}{4} - \cdots - \frac{20\pi}{4}$ $= -\frac{\pi}{4}(1+2+3+\cdots+20)$ $= -\frac{\pi}{4}\left(\frac{20}{2}(1+20)\right)$ $= -\frac{105\pi}{2}$ Since argument is between $-\pi$ to π $\therefore -\frac{\pi}{2}$	1 Mark
Q9	**A** $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$ $\overrightarrow{BA} = z_1 - z_2$ $\overrightarrow{MA} = \frac{1}{2}\overrightarrow{BA} = \frac{1}{2}(z_1 - z_2)$ $\overrightarrow{MQ} = i\overrightarrow{MA} = i\left(\frac{z_1 - z_2}{2}\right)$ $\overrightarrow{OQ} = \overrightarrow{OB} + \overrightarrow{BM} + \overrightarrow{MQ}$ $\overrightarrow{OQ} = z_2 + \frac{1}{2}(z_1 - z_2) + i\left(\frac{z_1 - z_2}{2}\right)$ $\overrightarrow{OQ} = \frac{2z_2 + z_1 - z_2}{2} + i\left(\frac{z_1 - z_2}{2}\right)$ $\overrightarrow{OQ} = \frac{z_1 + z_2}{2} + i\left(\frac{z_1 - z_2}{2}\right)$	1 Mark
Q10	**D** Option A is even, Option B is even, Option C is even. Only Option D is odd.	1 Mark

Section 2		
Q11a	zw^3 $= 2e^{\frac{i\pi}{4}} \times \left(3e^{\frac{5i\pi}{6}}\right)^3$ $= 2e^{\frac{i\pi}{4}} \times 27e^{\frac{5i\pi}{2}}$ $= 54e^{\frac{11i\pi}{4}}$ $= 54e^{\frac{3i\pi}{4}}$	2 Marks Correct 1 Mark Finds $w^3 = 27e^{\frac{5i\pi}{2}}$
Q11b	Sum of roots $\sqrt{5}\,\text{cis}\left(\frac{\pi}{6}\right) + \sqrt{5}\,\text{cis}\left(-\frac{\pi}{6}\right) = 2\sqrt{5}\cos\left(\frac{\pi}{6}\right) = 2\sqrt{5} \times \frac{\sqrt{3}}{2} = \sqrt{15}$ Product of roots $\sqrt{5}\,\text{cis}\left(\frac{\pi}{6}\right) \cdot \sqrt{5}\,\text{cis}\left(-\frac{\pi}{6}\right) = 5$ $\therefore x^2 - \sqrt{15}x + 5 = 0$	2 Marks Correct solution 1 Mark Determine the sum or product of roots
Q11c	$\overrightarrow{AB} = \begin{pmatrix} -9 \\ 7 \\ 5 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \\ 6 \end{pmatrix}$ $\therefore \underset{\sim}{r} = \begin{pmatrix} -4 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 4 \\ 6 \end{pmatrix}, \lambda \in \mathbb{R}$	2 Marks Correct solution 1 Mark Find $\overrightarrow{AB}$
Q11di	$\int_0^1 \frac{e^x}{e^{2x}+9}\,dx$ $= \left[\frac{1}{3}\tan^{-1}\frac{e^x}{3}\right]_0^1$ $= \frac{1}{3}\tan^{-1}\frac{e^1}{3} - \frac{1}{3}\tan^{-1}\frac{e^0}{3}$ $= \frac{1}{3}\left(\tan^{-1}\frac{e}{3} - \tan^{-1}\frac{1}{3}\right)$	2 Marks Correct solution 1 Mark Correct anti-derivative
Q11dii	$\frac{6x-4}{(x+1)(3x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{3x^2+2}$ $A(3x^2+2) + (Bx+C)(x+1) = 6x-4$ Let $x = -1$ $5A = -10$ $A = -2$ Let $x = 0$ $2A + C = -4$ $C = 0$ Let $x = 1$ $5A + 2(B+C) = 2$ $-10 + 2B = 2$ $B = 6$ $\frac{6x-4}{(x+1)(3x^2+2)} = \frac{-2}{x+1} + \frac{6x}{3x^2+2}$	4 Marks Correct solution 3 Marks Makes significant progress 2 Marks Correct partial fraction 1 Mark Correct A, B or C

	$\int_0^2 \frac{6x-4}{(x+1)(3x^2+2)}dx$ $= \int_0^2 \left(\frac{6x}{3x^2+2} - \frac{2}{x+1}\right) dx$ $= [\log_e\lvert 3x^2+2\rvert - 2\log_e\lvert x+1\rvert]_0^2$ $= [\log_e\lvert 3\times 2^2+2\rvert - 2\log_e\lvert 2+1\rvert] - [\log_e\lvert 3\times 0^2+2\rvert - 2\log_e\lvert 0+1\rvert]$ $= \log_e 14 - 2\log_e 3 - \log_e 2$ $= \log_e \frac{7}{9}$	
Q11diii	$\int_0^5 \sqrt{\frac{10-x}{10+x}}dx$ $= \int_0^5 \sqrt{\frac{10-x}{10+x}} \times \sqrt{\frac{10-x}{10-x}}dx$ $= \int_0^5 \frac{10-x}{\sqrt{100-x^2}}dx$ $= \int_0^5 \left(\frac{10}{\sqrt{100-x^2}} - \frac{x}{\sqrt{100-x^2}}\right)dx$ $= \left[10\sin^{-1}\frac{x}{10} + \sqrt{100-x^2}\right]_0^5$ $= \left(10\sin^{-1}\frac{5}{10} + \sqrt{100-5^2}\right) - \left(10\sin^{-1}0 + \sqrt{100-0^2}\right)$ $= \left(10\times\frac{\pi}{6} + \sqrt{75}\right) - (0+10)$ $= \frac{5\pi}{3} + 5\sqrt{3} - 10$	3 Marks Correct solution 2 Marks Correct anti-derivative 1 Mark Correct integration for either $\frac{10}{\sqrt{100-x^2}}$ or $\frac{x}{\sqrt{100-x^2}}$
Q12a	$\ddot{x} = -36(x+8)$ $\ddot{x} = -6^2(x+8)$ $n = 6, c = -8$ $T = \frac{2\pi}{6} = \frac{\pi}{3}$	2 Marks Correct solution 1 Mark Correct period or centre
Q12b	$\left(2\underset{\sim}{i} + m\underset{\sim}{j} - 3\underset{\sim}{k}\right)\cdot\left(m^2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}\right) = 0$ $2m^2 - m - 3 = 0$ $(2m-3)(m+1) = 0$ $m = \frac{3}{2}, m = -1$	2 Marks Correct solution 1 Mark Finds $2m^2 - m - 3 = 0$

Q12ci	$\forall x \in \mathbb{R},\ \ (x \geq x^2) \Rightarrow (x \leq 1)$	1 Mark Correct solution
Q12cii	$\exists x \in \mathbb{R}$ such that $(x > 1)$ and $(x \geq x^2)$ $\exists x \in \mathbb{R}$, $(x > 1) \cap (x \geq x^2)$	1 Mark Correct solution
Q12d	$\dfrac{dv}{dt} = \dfrac{v}{\log_e v}$ $\displaystyle\int \frac{\log_e v}{v}\,dv = \int dt$ $\dfrac{(\log_e v)^2}{2} = t + C$ $t = 0, v = 5$ $\dfrac{(\log_e 5)^2}{2} = 0 + C$ $C = \dfrac{(\log_e 5)^2}{2}$ $\dfrac{(\log_e v)^2}{2} = t + \dfrac{(\log_e 5)^2}{2}$ $\log_e v = \pm\sqrt{2t + (\log_e 5)^2}$ $v = e^{\pm\sqrt{2t+(\log_e 5)^2}}$ $\therefore\ v = e^{\sqrt{2t+(\log_e 5)^2}}$ as $t = 0, v = 5$	3 Marks Correct solution 2 Marks Correct anti-derivative 1 Mark Identifies $\displaystyle\int \frac{\log_e v}{v}\,dv = \int dt$
Q12e	$\displaystyle\int \frac{d\theta}{2 + 2\sin\theta + 2\cos\theta}$ $\displaystyle= \frac{1}{2}\int \frac{1}{\left(1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}\right)} \times \frac{2dt}{1+t^2}$ $\displaystyle= \frac{1}{2}\int \frac{2dt}{(1 + t^2 + 2t + 1 - t^2)}$ $\displaystyle= \frac{1}{2}\int \frac{2dt}{2 + 2t}$ $\displaystyle= \frac{1}{2}\int \frac{dt}{1 + t}$ $= \dfrac{1}{2}\log_e\lvert 1 + t\rvert + C$ $= \dfrac{1}{2}\log_e\left\lvert 1 + \tan\dfrac{\theta}{2}\right\rvert + C$ Or $= \log_e\sqrt{1 + \tan\dfrac{\theta}{2}} + C$	3 Marks Correct solution 2 Marks Correct anti-derivative 1 Mark Correct substitution of half angle formula

Q12f	Prove by contradiction: Assume $\sqrt[3]{p}$ is rational $\sqrt[3]{p} = \frac{m}{n}$ Where m and n are positive integers, $n \neq 0$, and m and n have no common factors. $p = \frac{m^3}{n^3}$ $n^3 p = m^3$ This means $n^3 p$ is divisible by m, but n is not divisible by m, so p is divisible by m. If $m = 1, p = \frac{1}{n^3}$ This is not possible as $p \geq 2$. If $m \geq 2$, where $p \neq m$ Since p is prime, p cannot be divisible by m If $m \geq 2$, where $p = m$ Then $n^3 m = m^3$, and $n^3 = m^2$, and this indicates that m and n have common factors, but this is a contradiction. $\therefore \sqrt[3]{p}$ is irrational.	3 Marks Correct solution 2 Marks Makes significant progress 1 Mark Establishes $\sqrt[3]{p} = \frac{m}{n}$ and conditions
Q13a	$P(x) = x^4 - 2x^3 - 14x^2 + 78x + 225$ $P'(x) = 4x^3 - 6x^2 - 28x + 78$ $P'(-3) = 4 \times (-3)^3 - 6 \times (-3)^2 - 28 \times (-3) + 78 = 0$ $P(-3) = (-3)^4 - 2 \times (-3)^3 - 14 \times (-3)^2 + 78 \times (-3) + 225 = 0$ $P(x) = (x+3)^2(ax^2 + bx + c)$ $P(x) = (x^2 + 6x + 9)(ax^2 + bx + c) = x^4 - 2x^3 - 14x^2 + 78x + 225$ Match coefficients $a = 1$ $9c = 225$ $c = 25$ $9b + 6c = 78$ $9b = 78 - 6 \times 25$ $b = -8$ $P(x) = (x+3)^2(x^2 - 8x + 25)$ $P(x) = (x+3)^2(x^2 - 8x + 16 + 9)$ $P(x) = (x+3)^2((x-4)^2 + 9)$ $P(x) = (x+3)^2((x-4)^2 - (3i)^2)$ $P(x) = (x+3)^2(x - 4 - 3i)(x - 4 + 3i)$ $(x+3)^2(x - 4 - 3i)(x - 4 + 3i) = 0$ $\therefore x = -3, x = 4 \pm 3i$	4 Marks Correct solution 3 Marks Determines $(x+3)^2((x-4)^2 - (3i)^2)$ 2 Marks Matches coefficients to find correct a and b or a and c 1 Mark Determines the double root at $x = -3$
Q13bi	$v^2 = 176x - 936 - 8x^2$ $\frac{1}{2}v^2 = 88x - 468 - 4x^2$ $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 88 - 8x$ $\therefore \ddot{x} = \frac{d^2x}{dt^2} = -8(x - 11)$	1 Mark Correct solution

Q13bii	$176x - 936 - 8x^2 = 0$ $x^2 - 22x - 117 = 0$ $(x-9)(x-13) = 0$ $x = 9, 13$ $\therefore$ amplitude is 2	2 Marks Correct solution 1 Mark Obtains $x = 9, 13$
Q13ci	$I = \int_0^a f(x)dx$ Let $x = a - u$ $dx = -du$ $x = a, u = 0$ $x = 0, u = a$ $I = \int_a^0 f(a-u)(-du)$ $I = \int_0^a f(a-u)du$ $I = \int_0^a f(a-x)dx$	1 Mark Correct solution
Q13cii	$\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x} dx$ $= \int_0^{\frac{\pi}{2}} \frac{\sin^m\left(\frac{\pi}{2} - x\right)}{\sin^m\left(\frac{\pi}{2} - x\right) + \cos^m\left(\frac{\pi}{2} - x\right)} dx$ $= \int_0^{\frac{\pi}{2}} \frac{\cos^m x}{\cos^m x + \sin^m x} dx$ $\int_0^{\frac{\pi}{2}} \frac{\cos^m x}{\cos^m x + \sin^m x} dx + \int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x} dx$ $= \int_0^{\frac{\pi}{2}} 1dx$ $= [x]_0^{\frac{\pi}{2}}$ $= \frac{\pi}{2} - 0$ $= \frac{\pi}{2}$ $\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^m x}{\cos^m x + \sin^m x} dx$ $\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1dx$ $\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x} dx = \frac{1}{2} \times \frac{\pi}{2}$ $\int_0^{\frac{\pi}{2}} \frac{\sin^m x}{\sin^m x + \cos^m x} dx = \frac{\pi}{4}$	3 Marks Correct solution 2 Marks Makes significant progress 1 Mark Obtains $\int_0^{\frac{\pi}{2}} \frac{\cos^m x}{\cos^m x + \sin^m x} dx$

Q13di	$\underset{\sim}{r_1} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k} + \lambda\left(\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}\right)$ $\underset{\sim}{r_2} = 8\underset{\sim}{i} + 8\underset{\sim}{j} + 13\underset{\sim}{k} + \mu\left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}\right)$ $4 + \lambda = 8 + 2\mu \quad (1)$ $3 + 4\lambda = 8 - 3\mu \quad (2)$ $1 + 3\lambda = 13 + 6\mu \quad (3)$ $(1) \times 4$ $16 + 4\lambda = 32 + 8\mu \quad (4)$ $(4) - (2)$ $13 = 24 + 11\mu$ $-11 = 11\mu$ $\mu = -1$ Sub into (1) $4 + \lambda = 8 + 2 \times -1$ $\lambda = 2$ Check using (3) $LHS = 1 + 3 \times 2 = 7$ $RHS = 13 + 6 \times -1 = 7$ $P = (4 + 2 \times 1, 3 + 4 \times 2, 1 + 2 \times 3) = (6, 11, 7)$	2 Marks Correct solution 1 Mark Determines the correct value of λ or μ
Q13dii	$\cos\theta = \dfrac{\left(\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k}\right) \cdot \left(2\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}\right)}{\sqrt{1^2 + 4^2 + 3^2} \times \sqrt{2^2 + (-3)^2 + 6^2}}$ $\cos\theta = \dfrac{1 \times 2 + 4 \times (-3) + 3 \times 6}{\sqrt{26} \times \sqrt{49}}$ $\cos\theta = \dfrac{8}{7\sqrt{26}}$ $\theta = 77°2'53.22''$ $\theta = 77°3'$ (nearest minute)	2 Marks Correct solution 1 Mark Makes significant progress
Q14a	$\int \dfrac{x + 8}{\sqrt{24 - 10x - x^2}} dx$ $= \int \dfrac{x + 5}{\sqrt{24 - 10x - x^2}} dx + \int \dfrac{3}{\sqrt{24 - 10x - x^2}} dx$ $= -\int \dfrac{-(2x + 10)(24 - 10x - x^2)^{-\frac{1}{2}}}{2} dx + \int \dfrac{3}{\sqrt{49 - (x^2 + 10x + 25)}} dx$ $= -\int \dfrac{-(2x + 10)(24 - 10x - x^2)^{-\frac{1}{2}}}{2} dx + \int \dfrac{3}{\sqrt{7^2 - (x + 5)^2}} dx$ $= -\sqrt{24 - 10x - x^2} + 3\sin^{-1}\left(\dfrac{x + 5}{7}\right) + C$	3 Marks Correct solution 2 Marks Makes significant progress 1 Mark Correct splitting into two fractions

Q14b	Let the 3 consecutive numbers be $n, n+1, n+2$ $n^2+(n+1)^2+(n+2)^2$ $= n^2+n^2+2n+1+n^2+4n+4$ $= 3n^2+6n+5$ $= 3n^2+6n+6-1$ $= 3(n^2+2n+3)-1$ ∴ This is one less than a multiple of 3.	2 Marks Correct solution 1 Mark Obtains $3n^2+6n+5$ or equivalence
Q14c	$I=\int_1^2 \frac{\log_e x}{x^3}\,dx$ $u=\log_e x \qquad v'=x^{-3}$ $u'=\frac{1}{x} \qquad v=\frac{x^{-2}}{-2}=-\frac{1}{2x^2}$ $I=\left[-\frac{\log_e x}{2x^2}\right]_1^2-\int_1^2 -\frac{1}{2x^3}\,dx$ $I=\left[-\frac{\log_e x}{2x^2}\right]_1^2+\frac{1}{2}\int_1^2 x^{-3}\,dx$ $I=\left[\left(-\frac{\log_e 2}{2\times 2^2}\right)-\left(-\frac{\log_e 1}{2\times 1^2}\right)\right]+\frac{1}{2}\left[\frac{x^{-2}}{2}\right]_1^2$ $I=-\frac{\log_e 2}{8}+\frac{1}{2}\left[\left(-\frac{1}{2\times 2^2}\right)-\left(-\frac{1}{2\times 1^2}\right)\right]$ $I=-\frac{\log_e 2}{8}+\frac{1}{2}\left(-\frac{1}{8}+\frac{1}{2}\right)$ $I=\frac{3}{16}-\frac{\log_e 2}{8}$	4 Marks Correct solution 3 Marks Correct anti-derivative 2 Marks Correctly uses parts 1 Mark Correct u, u', v, v' and attempts to use parts
Q14di	$x \geq 0, y \geq 0$ $\left(\sqrt{x}-\sqrt{y}\right)^2 \geq 0$ $x-2\sqrt{xy}+y \geq 0$ $x+y \geq 2\sqrt{xy}$ $\frac{x+y}{2} \geq \sqrt{xy}$	1 Mark Correct solution
Q14dii	$a^8+b^8+c^8=\frac{1}{2}(a^8+b^8)+\frac{1}{2}(b^8+c^8)+\frac{1}{2}(a^8+c^8)$ $\frac{1}{2}(a^8+b^8) \geq \sqrt{a^8b^8}$ $\frac{1}{2}(b^8+c^8) \geq \sqrt{b^8c^8}$ $\frac{1}{2}(a^8+c^8) \geq \sqrt{a^8c^8}$ $a^8+b^8+c^8 \geq \sqrt{a^8b^8}+\sqrt{b^8c^8}+\sqrt{a^8c^8}$ $a^8+b^8+c^8 \geq a^4b^4+b^4c^4+a^4c^4$	2 Marks Correct solution 1 Mark Makes significant progress

Q14diii	$a^4+b^4 \geq 2a^2b^2 \quad (1)$ $b^4+c^4 \geq 2b^2c^2 \quad (2)$ $a^4+c^4 \geq 2a^2c^2 \quad (3)$ $(1)+(2)+(3)$ $2(a^4+b^4+c^4) \geq 2(a^2b^2+b^2c^2+a^2c^2)$ $a^4+b^4+c^4 \geq a^2b^2+b^2c^2+a^2c^2$ Replacing a by ab, b by bc, c by ac $(ab)^4+(bc)^4+(ac)^4 \geq (ab)^2(bc)^2+(bc)^2(ac)^2+(ab)^2(ac)^2$ $a^4b^4+b^4c^4+a^4c^4 \geq a^2b^4c^2+a^2b^2c^4+a^4b^2c^2$ $\therefore a^4b^4+b^4c^4+a^4c^4 \geq a^4b^2c^2+a^2b^4c^2+a^2b^2c^4$	2 Marks Correct solution 1 Mark Shows $a^4+b^4+c^4$ $\geq a^2b^2+b^2c^2+a^2c^2$
Q14div	$a^8+b^8+c^8 \geq a^4b^4+b^4c^4+a^4c^4 \geq a^4b^2c^2+a^2b^4c^2+a^2b^2c^4$ $a^8+b^8+c^8 \geq a^4b^2c^2+a^2b^4c^2+a^2b^2c^4$ $a^8+b^8+c^8 \geq a^2b^2c^2(a^2+b^2+c^2)$ If $a^2+b^2+c^2=d^2$, then $a^8+b^8+c^8 \geq a^2b^2c^2d^2$	1 Mark Correct solution
Q15a	$x=8v-3\ln(v+2)$ $\frac{dx}{dv}=8-\frac{3}{v+2}$ $\frac{dx}{dv}=\frac{8v+16-3}{v+2}$ $\frac{dx}{dv}=\frac{8v+13}{v+2}$ $\frac{dv}{dt}=v\frac{dv}{dx}$ $\frac{dv}{dt}=v\left(\frac{v+2}{8v+13}\right)$ $\int_1^v \frac{8v+13}{v(v+2)}dv=\int_0^t dt$ $\frac{8v+13}{v(v+2)}=\frac{A}{v}+\frac{B}{v+2}$ $A(v+2)+Bv=8v+13$ $v=0$ $2A=13$ $A=\frac{13}{2}$ $v=-2$ $-2B=-16+13$ $B=\frac{3}{2}$ $\frac{A}{v}+\frac{B}{v+2}=\frac{13}{2v}+\frac{3}{2v+4}$	4 Marks Correct solution 3 Marks Correct anti-derivative 2 Marks Correct partial fraction 1 Mark Finds $\frac{dx}{dv}=\frac{8v+13}{v+2}$

	$\int_0^t dt = \int_1^v \left(\frac{13}{2v} + \frac{3}{2(v+2)}\right) dv$ $t = \left[\frac{13}{2}\ln\lvert v\rvert + \frac{3}{2}\ln\lvert v+2\rvert\right]_1^v$ $t = \left[\frac{13}{2}\ln\lvert v\rvert + \frac{3}{2}\ln\lvert v+2\rvert - \left(\frac{13}{2}\ln 1 + \frac{3}{2}\ln 3\right)\right]$ $t = \frac{13}{2}\ln\lvert v\rvert + \frac{3}{2}\ln\left\lvert\frac{v+2}{3}\right\rvert$	
Q15bi	$I_n = \int_0^1 x^{1000}(1-x)^n dx$ $u = (1-x)^n, \qquad v' = x^{1000}$ $u' = -n(1-x)^{n-1}, \qquad v = \frac{x^{1001}}{1001}$ $I_n = \left[\frac{x^{1001}}{1001}(1-x)^n\right]_0^1 - \int_0^1 \frac{x^{1001}}{1001}(-n(1-x)^{n-1})dx$ $I_n = \left[\frac{1^{1001}}{1001}(1-1)^n - \frac{0^{1001}}{1001}(1-0)^n\right] + \frac{n}{1001}\int_0^1 x^{1001}(1-x)^{n-1}dx$ $I_n = \frac{n}{1001}\int_0^1 x^{1001}(1-x)^{n-1}dx$	2 Marks Correct solution 1 Mark Correctly uses parts
Q15bii	$I_n = \int_0^1 x^{1000}(1-x)^n dx$ $I_n = \frac{n}{1001}\int_0^1 x^{1000} \times x \times (1-x)^{n-1}dx$ $I_n = -\frac{n}{1001}\int_0^1 x^{1000} \times \big((1-x)-1\big) \times (1-x)^{n-1}dx$ $I_n = -\frac{n}{1001}\int_0^1 (x^{1000}(1-x)(1-x)^{n-1} - x^{1000}(1-x)^{n-1})\, dx$ $I_n = -\frac{n}{1001}\left(\int_0^1 x^{1000}(1-x)^n dx - \int_0^1 x^{1000}(1-x)^{n-1}dx\right)$ $I_n = -\frac{n}{1001}(I_n - I_{n-1})$ $I_n + \frac{n}{1001}I_n = \frac{n}{1001}I_{n-1}$ $I_n\left(1 + \frac{n}{1001}\right) = \frac{n}{1001}I_{n-1}$ $I_n\left(\frac{1001+n}{1001}\right) = \frac{n}{1001}I_{n-1}$ $I_n = \frac{n}{1001}I_{n-1} \times \frac{1001}{1001+n}$ $\therefore I_n = \frac{n}{1001+n}I_{n-1}$	2 Marks Correct solution 1 Mark Makes significant progress

Q15biii	$I_n = \frac{n}{1001+n} I_{n-1}$ $I_n = \frac{n}{1001+n} \times \frac{n-1}{1001+n-1} \times \frac{n-2}{1001+n-2} \times \ldots \times \frac{1}{1001+1} \times I_0$ $I_0 = \int_0^1 x^{1000}(1-x)^0 dx$ $I_0 = \left[\frac{x^{1001}}{1001}\right]_0^1$ $I_0 = \frac{1^{1001}}{1001} - \frac{0^{1001}}{1001}$ $I_0 = \frac{1}{1001}$ $I_n = \frac{n}{1001+n} \times \frac{n-1}{1001+n-1} \times \frac{n-2}{1001+n-2} \times \ldots \times \frac{1}{1001+1} \times \frac{1}{1001}$ $I_n = \frac{n}{1001+n} \times \frac{n-1}{1000+n} \times \frac{n-2}{999+n} \times \ldots \times \frac{1}{1002} \times \frac{1}{1001}$ $I_n = \frac{n}{1001+n} \times \frac{n-1}{1000+n} \times \frac{n-2}{999+n} \times \ldots \times \frac{1}{1002} \times \frac{1}{1001} \times \frac{1000!}{1000!}$ $I_n = \frac{n!\,1000!}{(1001+n)!}$	2 Marks Correct solution 1 Mark Expands I_n and finds I_0
Q15ci	$f(x) = \frac{(n+1+x)^{n+1}}{(n+x)^n}, x \geq 0, n \geq 1$ $u = (n+1+x)^{n+1}$ $\quad v = (n+x)^n$ $u' = (n+1)(n+1+x)^n$ $\quad v' = n(n+x)^{n-1}$ $f'(x) = \frac{(n+x)^n(n+1)(n+1+x)^n - (n+1+x)^{n+1}n(n+x)^{n-1}}{[(n+x)^n]^2}$ $f'(x) = \frac{(n+1+x)^n(n+x)^{n-1}[(n+x)(n+1) - n(n+1+x)]}{(n+x)^{2n}}$ $f'(x) = \frac{(n+1+x)^n[n^2+n+xn+x-n^2-n-nx]}{(n+x)^{(2n-n+1)}}$ $f'(x) = \frac{x(n+1+x)^n}{(n+x)^{(n+1)}}$ Since $x \geq 0, n \geq 1$ $(n+1+x)^n > 0, (n+x)^{(n+1)} > 0$ $\therefore f'(x) > 0$ $\therefore f(x)$ is increasing for $x > 0$.	2 Marks Correct solution 1 Mark Correct differentiation and simplifies

Q15cii	Since $f(x)$ is increasing for $x>0$ Then $f(x)>f(0)$ $\frac{(n+1+x)^{n+1}}{(n+x)^n}>\frac{(n+1+0)^{n+1}}{(n+0)^n}$ $\frac{(n+1+x)^{n+1}}{(n+x)^n}>\frac{(n+1)^{n+1}}{(n)^n}$ Since $x>0$, $(n+x)^n>0$, $(n+1)^{n+1}>0$ $\frac{(n+1+x)^{n+1}}{(n+1)^{n+1}}>\frac{(n+x)^n}{(n)^n}$ ……(1) $\left(\frac{n+1+x}{n+1}\right)^{n+1}>\left(\frac{n+x}{n}\right)^n$ $\therefore \left(1+\frac{x}{n+1}\right)^{n+1}>\left(1+\frac{x}{n}\right)^n$	2 Marks Correct solution 1 Mark Explains $f(x)>f(0)$ and shows $\frac{(n+1+x)^{n+1}}{(n+x)^n}>\frac{(n+1)^{n+1}}{(n)^n}$
Q15ciii	Sub $x=1$ into (1) $\frac{(n+1+1)^{n+1}}{(n+1)^{n+1}}>\frac{(n+1)^n}{(n)^n}$ $\frac{(n+2)^{n+1}}{(n+1)^{n+1}}>\frac{(n+1)^n}{(n)^n}$ Since $(n+1)^n>0$, $(n+1)^{n+1}>0$ $(n+2)^{n+1}n^n>(n+1)^n(n+1)^{n+1}$ $\therefore (n+2)^{n+1}n^n>(n+1)^{2n+1}$	1 Mark Correct solution
Q16ai	$x^9-1=(ax^3-bx^2+cx+d)(x^6+x^3+1)$ Compare coefficients $a=1, d=-1, b=0, c=0$ $x^9-1=(x^3-1)(x^6+x^3+1)$ The roots of $x^9-1=0$ are the roots of $x^3-1=0$ or the roots of $x^6+x^3+1=0$. $\therefore$ The roots of $P(x)=0$ are amongst the roots of $x^9-1=0$.	1 Mark Correct solution
Q16aii	$x^9-1=0$ $x^9=1$ $(\cos 9\theta+i\sin 9\theta)=1$ $9\theta=0+2k\pi \quad (k=0,\pm1,\pm2,\dots)$ $\theta=\frac{2k\pi}{9}$ $x=\left(\cos\frac{2k\pi}{9}+i\sin\frac{2k\pi}{9}\right)$ $k=0, \quad \cos 0+i\sin 0=\cos 0=1$ $k=1, \quad \cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}$	4 Marks Correct solution 3 Marks Determine roots of $x^6+x^3+1=0$ 2 Marks Finds all solutions to $x^9=1$ 1 Mark Finds some solutions to $x^9=1$

	$k=-1,\ \cos\left(-\frac{2\pi}{9}\right)+i\sin\left(-\frac{2\pi}{9}\right)$ $k=2,\ \cos\frac{4\pi}{9}+i\sin\frac{4\pi}{9}$ $k=-2,\ \cos\left(-\frac{4\pi}{9}\right)+i\sin\left(-\frac{4\pi}{9}\right)$ $k=3,\ \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$ $k=-3,\ \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$ $k=4,\ \cos\frac{8\pi}{9}+i\sin\frac{8\pi}{9}$ $k=-4,\ \cos\left(-\frac{8\pi}{9}\right)+i\sin\left(-\frac{8\pi}{9}\right)$ The roots of $x^3-1=0$ are: $1, \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}, \cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3}$ So the roots of $x^6+x^3+1=0$ are: $\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}, \cos\left(-\frac{2\pi}{9}\right)+i\sin\left(-\frac{2\pi}{9}\right), \cos\frac{4\pi}{9}+i\sin\frac{4\pi}{9},$ $\cos\left(-\frac{4\pi}{9}\right)+i\sin\left(-\frac{4\pi}{9}\right), \cos\frac{8\pi}{9}+i\sin\frac{8\pi}{9}, \cos\left(-\frac{8\pi}{9}\right)+i\sin\left(-\frac{8\pi}{9}\right)$ $\alpha=\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}$ $\bar{\alpha}=\cos\left(-\frac{2\pi}{9}\right)+i\sin\left(-\frac{2\pi}{9}\right)=\cos\frac{2\pi}{9}-i\sin\frac{2\pi}{9}$ $\beta=\cos\frac{4\pi}{9}+i\sin\frac{4\pi}{9}$ $\bar{\beta}=\cos\left(-\frac{4\pi}{9}\right)+i\sin\left(-\frac{4\pi}{9}\right)=\cos\frac{4\pi}{9}-i\sin\frac{4\pi}{9}$ $\gamma=\cos\frac{8\pi}{9}+i\sin\frac{8\pi}{9}$ $\bar{\gamma}=\cos\left(-\frac{8\pi}{9}\right)+i\sin\left(-\frac{8\pi}{9}\right)=\cos\frac{8\pi}{9}-i\sin\frac{8\pi}{9}$ $P(x)=(x-\alpha)(x-\bar{\alpha})(x-\beta)(x-\bar{\beta})(x-\gamma)(x-\bar{\gamma})$ $P(x)=(x^2-2Re(\alpha)x+1)\left((x^2-2Re(\beta)x+1)\right)\left((x^2-2Re(\gamma)x+1)\right)$ $P(x)=\left(x^2-2\cos\frac{2\pi}{9}x+1\right)\left(\left(x^2-2\cos\frac{4\pi}{9}x+1\right)\right)\left(\left(x^2-2\cos\frac{8\pi}{9}x+1\right)\right)$	
16aiii	Equating coefficient of x^2 on both sides of the equation $0=1+1+1+\left(-2\cos\frac{2\pi}{9}\right)\times\left(-2\cos\frac{4\pi}{9}\right)+\left(-2\cos\frac{4\pi}{9}\right)\times\left(-2\cos\frac{8\pi}{9}\right)+\left(-2\cos\frac{2\pi}{9}\right)\times\left(-2\cos\frac{8\pi}{9}\right)$ $-3=4\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}+4\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}+4\cos\frac{2\pi}{9}\cos\frac{8\pi}{9}$ $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}+\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}+\cos\frac{2\pi}{9}\cos\frac{8\pi}{9}=-\frac{3}{4}$	2 Marks Correct solution 1 Mark Attempts to equate coefficient of x^2

Q16bi	$\|f(x)\|$ is always positive so $\int_a^b \|f(x)\|dx$ is a positive value. $f(x)$ can cross the x-axis for $a \le x \le b$, then $\int_a^b f(x)dx$ may involve adding negative (below x-axis) and positive values (above x-axis). So $\left\|\int_a^b f(x)dx\right\| < \int_a^b \|f(x)\|dx$ $\left\|\int_a^b f(x)dx\right\| = \int_a^b \|f(x)\|dx$ if $f(x)$ is positive. $\therefore \left\|\int_a^b f(x)dx\right\| \le \int_a^b \|f(x)\|dx$	1 Mark Correct solution
Q16bii	$\left\|\int_0^\pi 27^x \cos x\, dx\right\| \le \int_0^\pi \|27^x \cos x\|dx$ $\left\|\int_0^\pi 27^x \cos x\, dx\right\| \le \int_0^\pi 27^x dx \qquad (\|\cos x\| \le 1, 27^x > 0)$ $\left\|\int_0^\pi 27^x \cos x\, dx\right\| \le \left[\frac{27^x}{\ln 27}\right]_0^\pi$ $\left\|\int_0^\pi 27^x \cos x\, dx\right\| \le \left[\frac{3^{3x}}{3\ln 3}\right]_0^\pi$ $\left\|\int_0^\pi 27^x \cos x\, dx\right\| \le \frac{3^{3\pi}}{3\ln 3} - \frac{3^{3\times 0}}{3\ln 3}$ $\left\|\int_0^\pi 27^x \cos x\, dx\right\| \le \frac{3^{3\pi} - 1}{3\ln 3}$	3 Marks Correct solution 2 Marks Correct anti-derivative 1 Mark Obtains $\left\|\int_0^\pi 27^x \cos x\, dx\right\|$ $\le \int_0^\pi 27^x dx$
Q16c	$RTP\colon \ln(x_1 + x_2 + \cdots + x_n) > \frac{1}{2^{n-1}}(\ln x_1 + \ln x_2 + \cdots + \ln x_n)$ 1. Prove statement is true for $n = 2$ $LHS = \ln(x_1 + x_2) \qquad RHS = \frac{1}{2^{2-1}}(\ln x_1 + \ln x_2) = \frac{1}{2}(\ln x_1 + \ln x_2)$ Since $x_1 + x_2 > \sqrt{x_1 x_2}$, and $\ln x$is an increasing function, $\ln(x_1 + x_2) > \ln\left(\sqrt{x_1 x_2}\right)$ $\ln(x_1 + x_2) > \frac{1}{2}\ln(x_1 x_2)$ $\ln(x_1 + x_2) > \frac{1}{2}(\ln x_1 + \ln x_2)$ $LHS > RHS$ $\therefore$ This is true for $n = 2$. 2. Assume statement is true for $n = k, k \in \mathbb{Z}^+$ $\ln(x_1 + x_2 + \cdots + x_k) > \frac{1}{2^{k-1}}(\ln x_1 + \ln x_2 + \cdots + \ln x_k)$	4 Marks Correct solution 3 Marks Makes significant progress 2 Marks Shows $\ln((x_1 + x_2 + \cdots + x_k) + x_{k+1})$ $> \frac{1}{2}(\ln(x_1 + x_2 + \cdots + x_k)$ $+ \ln x_{k+1})$ 1 Mark Proves the initial case

	3. Prove statement is true for $n = k + 1$ $\ln(x_1 + x_2 + \cdots + x_k + x_{k+1}) > \frac{1}{2^k}(\ln x_1 + \ln x_2 + \cdots + \ln x_k + \ln x_{k+1})$ $LHS = \ln\big((x_1 + x_2 + \cdots + x_k) + x_{k+1}\big)$ $\ln\big((x_1 + x_2 + \cdots + x_k) + x_{k+1}\big) > \frac{1}{2}(\ln(x_1 + x_2 + \cdots + x_k) + \ln x_{k+1})$ From assumption $\ln\big((x_1 + x_2 + \cdots + x_k) + x_{k+1}\big)$ $> \frac{1}{2}\left(\frac{1}{2^{k-1}}(\ln x_1 + \ln x_2 + \cdots + \ln x_k) + \ln x_{k+1}\right)$ $\ln(x_1 + x_2 + \cdots + x_k + x_{k+1}) > \frac{1}{2^k}(\ln x_1 + \ln x_2 + \cdots + \ln x_k) + \frac{1}{2}\ln x_{k+1}$ Since $\frac{1}{2} > \frac{1}{2^k}$ and $\ln x_{k+1} > 0$, then $\frac{1}{2^k}(\ln x_1 + \ln x_2 + \cdots + \ln x_k) + \frac{1}{2}\ln x_{k+1}$ $> \frac{1}{2^k}(\ln x_1 + \ln x_2 + \cdots + \ln x_k) + \frac{1}{2^k}\ln x_{k+1}$ $\frac{1}{2^k}(\ln x_1 + \ln x_2 + \cdots + \ln x_k) + \frac{1}{2}\ln x_{k+1}$ $> \frac{1}{2^k}(\ln x_1 + \ln x_2 + \cdots + \ln x_k + \ln x_{k+1})$ $\therefore \ln(x_1 + x_2 + \cdots + x_k + x_{k+1}) > \frac{1}{2^k}(\ln x_1 + \ln x_2 + \cdots + \ln x_k + \ln x_{k+1})$ True by mathematical induction for all integers $n \geq 2$.	